

QUASI-INVARIANCE PROPERTIES OF A CLASS OF SUBORDINATORS

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ABSTRACT. We study absolute-continuity properties of a class of stochastic processes, including the gamma and the Dirichlet processes. We prove that the laws of a general class of non-linear transformations of such processes are locally equivalent to the law of the original process and we compute explicitly the associated Radon-Nikodym densities. This work unifies and generalizes to random non-linear transformations several previous results on quasi-invariance of gamma and Dirichlet processes.

1. INTRODUCTION

In this paper we present several absolute-continuity results concerning, among others, the gamma process and the Dirichlet processes. We recall that the gamma process $(\gamma_t)_{t \geq 0}$ is a subordinator, i.e. a non-decreasing Lévy process, with gamma marginals, i.e. $\gamma_0 = 0$ and

$$\mathbb{P}(\gamma_t \in dx) = p_t(x)dx, \quad p_t(x) := 1_{[0, \infty)}(x) \frac{1}{\Gamma(t)} x^{t-1} e^{-x}, \quad t > 0, x \in \mathbb{R}.$$

Moreover for any $T > 0$, we define the Dirichlet process over $[0, T]$ as $D_t^{(T)} := \gamma_t/\gamma_T$, $t \in [0, T]$; we recall that γ_T is independent of $(\gamma_t/\gamma_T, t \in [0, T])$ and that, therefore, $(D_t^{(T)}, t \in [0, T])$ is equal in law to the gamma process conditioned on $\{\gamma_T = 1\}$. See [11] for a survey of the main properties of the gamma process.

The gamma process has been the object of intense research activity in recent years, both from pure and applied perspectives, such as in representation theory of infinite dimensional groups, in mathematical finance and in mathematical biology (see e.g. [9, 3, 5]). Quasi-invariance properties of the associated probability measure on path or measure space with respect to canonical transformations often play a central role. We recall that, given a measure μ on a space X and a measurable map $T : X \mapsto X$, quasi-invariance of μ under T means that μ and the image measure $T_*\mu$ are equivalent, i.e. mutually absolutely continuous. A classical example is the Girsanov formula for additive perturbations of Brownian motion (see, e.g., [7], Chap. VIII).

In this paper we study quasi-invariance properties for a class of subordinators which we denote by (\mathcal{L}) and define below, with respect to a large class of non-linear sample path transformations. In particular, we unify and extend previous results on the real valued gamma and Dirichlet processes.

Quasi-invariance properties of Lévy processes have been studied for quite some time, see e.g. Sato [8, p. 217-218]. In the case of the gamma process, for any measurable function $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with a and $1/a$ bounded, the laws of $(\int_0^t a_s d\gamma_s, t \geq 0)$ and $(\gamma_t, t \geq 0)$ are

locally equivalent, see [9]. By local equivalence of two real-valued processes $(\eta_t, t \geq 0)$ and $(\zeta_t, t \geq 0)$, we mean that for all $T > 0$ the laws of $(\eta_t, t \in [0, T])$ and $(\zeta_t, t \in [0, T])$ are equivalent.

Here, we show the same property for a much wider class of transformations $\xi \mapsto K(\xi)$, e.g. $(\xi_t)_{t \geq 0} \mapsto (K(t, \xi_t))_{t \geq 0}$ and $(\xi_t)_{t \geq 0} \mapsto (\sum_{s \leq t} K(s, \Delta \xi_s))_{t \geq 0}$, where $(\xi_t)_{t \geq 0}$ is a (\mathcal{L}) -subordinator and $K(s, \cdot)$ is a $C^{1,\alpha}$ -isomorphism of \mathbb{R}_+ for each $s \geq 0$ and $\alpha \in]0, 1[$. Using the mentioned properties of the gamma process, we establish analogous quasi-invariance results for transformations $D^{(T)} \mapsto K(D^{(T)})$ of the Dirichlet process, e.g. $(D_t^{(T)})_{t \in [0, T]} \mapsto (K(t, D_t^{(T)}))_{t \in [0, T]}$, where $K(s, \cdot)$ is an increasing $C^{1,\alpha}$ -isomorphism of $[0, 1]$ for each $s \in [0, T]$.

In all these cases, we compute the Radon-Nikodym density explicitly and study its martingale structure. We notice that our approach allows to treat the previously mentioned results by Vershik-Tsilevich-Yor [9, 10], together with Handa's [5] and the recent one by Renesse-Sturm [6] on Dirichlet processes, within a unified framework.

The paper ends with an application to SDEs driven by (\mathcal{L}) -subordinators. Finally we point out that, in the same spirit as in [6], each quasi-invariance property we show yields easily an integration by parts formula on the path space; such formulae can be used in order to study an appropriate Dirichlet form and the associated infinite-dimensional diffusion process. These applications will be developed in a future work.

1.1. The main result. Let $(\xi_t)_{t \geq 0}$ be a subordinator, i.e. an increasing Lévy process with $\xi_0 = 0$. In this paper we consider subordinators in the class (\mathcal{L}) , meaning *with logarithmic singularity*, i.e. we assume that ξ has zero drift and Lévy measure

$$\nu(dx) = g(x) dx, \quad x > 0,$$

where $g :]0, \infty[\mapsto \mathbb{R}_+$ is measurable and satisfies

$$(H1) \quad g > 0 \quad \text{and} \quad \int_1^\infty g(x) dx < \infty;$$

$$(H2) \quad \text{there exist } g_0 \geq 0 \text{ and } \zeta : [0, 1] \mapsto \mathbb{R} \text{ measurable such that}$$

$$g(x) = \frac{g_0}{x} + \zeta(x), \quad \forall x \in]0, 1], \quad \text{and} \quad \int_0^1 |\zeta(x)| dx < +\infty.$$

We recall that for all $t \geq 0$, $\lambda > 0$

$$\mathbb{E}(e^{-\lambda \xi_t}) = \exp(-t\Psi(\lambda)), \quad \Psi(\lambda) := \int_0^\infty (1 - e^{-\lambda x}) g(x) dx.$$

For the general theory of subordinators, see [4]. We denote by $\mathcal{F}_t := \sigma(\xi_s : s \leq t)$, $t \geq 0$, the filtration of ξ . We denote the space of càdlàg functions on $[0, t]$ by $\mathcal{D}([0, t])$, endowed with the Skorohod topology.

Remark 1.1. In the particular case of the gamma process $(\gamma_t)_{t \geq 0}$, mentioned above, we have

$$g(x) = \frac{e^{-x}}{x}, \quad x > 0, \quad \Psi(\lambda) = \log(1 + \lambda), \quad \lambda \geq 0.$$

We consider a measurable function $h : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

- (1) h is $\mathcal{P} \otimes \mathcal{B}_{\mathbb{R}_+}$ -measurable, where \mathcal{P} denotes the predictable σ -algebra generated by ξ ;
- (2) denoting $h(s, a) = h(s, \omega, a)$, there exist finite constants $\kappa > 1$ and $\alpha \in]0, 1[$, such that almost surely

$$|h(s, x) - h(s, y)| \leq \kappa |x - y|^\alpha, \quad \forall x, y \in \mathbb{R}_+, s \geq 0, \quad (1.1)$$

$$0 < \kappa^{-1} \leq h(s, x) \leq \kappa < \infty, \quad \forall x \in \mathbb{R}_+, s \geq 0. \quad (1.2)$$

Then we set

$$H(s, x) = \int_0^x h(s, y) dy, \quad \forall x \geq 0, s \geq 0.$$

Notice that a.s. $H(s, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is necessarily a C^1 -diffeomorphism for all $s \geq 0$. We set

$$\Delta \xi_s := \xi_s - \xi_{s-}, \quad s \geq 0,$$

and for convenience of notation

$$h(s, 0) \cdot \frac{g(H(s, 0))}{g(0)} := 1, \quad \forall s \geq 0. \quad (1.3)$$

We can now state the main result of this paper

Theorem 1.2.

- (1) *The process*

$$M_t^H := \exp \left(g_0 \int_0^t \log h(s, 0) ds \right) \prod_{s \leq t} \left[h(s, \Delta \xi_s) \cdot \frac{g(H(s, \Delta \xi_s))}{g(\Delta \xi_s)} \right], \quad t \geq 0,$$

is a $(\mathcal{F}_t, \mathbb{P})$ -martingale with $\mathbb{E}(M_t^H) = 1$ and a.s. $M_t^H > 0$. We can uniquely define a probability measure \mathbb{P}^H such that $\mathbb{P}_{|\mathcal{F}_t}^H = M_t^H \cdot \mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$.

- (2) *Setting*

$$\xi_t^H := \sum_{s \leq t} H(s, \Delta \xi_s), \quad t \geq 0, \quad (1.4)$$

then ξ^H is distributed under \mathbb{P}^H as ξ under \mathbb{P} .

Note that Theorem 1.2 is a local equivalence result for the laws of ξ and ξ^H , since a.s. $M_t^H > 0$. The theorem is stated for general subordinators in the class (\mathcal{L}) defined above and for a general random transformation; in section 3 we consider some special cases of the general result, and in section 4 we consider the case of the Dirichlet process.

1.2. A parallel between the gamma process and Brownian motion. The absolute-continuity results presented in this paper can be better understood by comparison with some analogous properties of Brownian motion.

The Girsanov theorem for a Brownian motion $(B_t, t \geq 0)$ states the following property: if $(a_s, s \geq 0)$ is an adapted and (say) bounded process, then the law of the process

$$t \mapsto B_t + \int_0^t a_s ds, \quad t \geq 0,$$

is locally equivalent to that of $(B_t, t \geq 0)$, with explicit Radon-Nikodym density. We call this property quasi-invariance by addition.

As a byproduct case of our Theorem 1.2, the gamma process γ has an analogous property of quasi-invariance by multiplication (see also [9]): if $(a_s, s \geq 0)$ is a predictable process such that a and $1/a$ are bounded, then the law of

$$t \mapsto \int_0^t a_s d\gamma_s, \quad t \geq 0,$$

is locally equivalent to that of $(\gamma_t, t \geq 0)$, and we compute explicitly the Radon-Nikodym density. In fact, we can prove the same quasi-invariance property for all (\mathcal{L}) -subordinators.

The Girsanov theorem for Brownian motion has important applications in the study of stochastic differential equations (SDEs) driven by a Wiener process; likewise, our Theorem 1.2 allows to give analogous applications to SDEs driven by (\mathcal{L}) -subordinators, e.g. to compute explicitly laws of solutions; see section 5.

2. A GENERALIZATION OF A FORMULA OF TSILEVICH-VERSHIK-YOR

Within the framework of subsection 1.1, the law of ξ with $\xi_0 = 0$ is characterized by its Laplace transform, i.e. for any measurable bounded $\lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$

$$\mathbb{E} \left[\exp \left(- \int_0^t \lambda_s d\xi_s \right) \right] = \exp \left(- \int_0^t \Psi(\lambda_s) ds \right).$$

In order to prove Theorem 1.2, we shall show that ξ^H has, under \mathbb{P}^H , the same Laplace transform as ξ under \mathbb{P} , namely for all measurable bounded $\lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$

$$\mathbb{E}^H \left[\exp \left(- \int_0^t \lambda_s d\xi_s^H \right) \right] = \exp \left(- \int_0^t \Psi(\lambda_s) ds \right).$$

To do that, we shall show that the process

$$\exp \left(- \int_0^t \lambda_s d\xi_s^H + \int_0^t \Psi(\lambda_s) ds \right), \quad t \geq 0,$$

is a $((\mathcal{F}_t), \mathbb{P}^H)$ -martingale, which is equivalent to prove the following

Proposition 2.1. *We set for all $t \geq 0$*

$$\begin{aligned} M_t^{H,\lambda} &:= \exp \left(\int_0^t (g_0 \log h(s, 0) + \Psi(\lambda_s)) ds \right) \cdot \\ &\cdot \prod_{s \leq t} \left[h(s, \Delta \xi_s) \frac{g(H(s, \Delta \xi_s))}{g(\Delta \xi_s)} \exp(-\lambda_s H(s, \Delta \xi_s)) \right]. \end{aligned} \tag{2.1}$$

Then $M^{H,\lambda}$ is a $(\mathcal{F}_t, \mathbb{P})$ -martingale with $\mathbb{E}(M_t^{H,\lambda}) = 1$ and a.s. $M_t^{H,\lambda} > 0$.

Tsilevich-Vershik-Yor prove in [9] the same result for ξ a gamma process and $H(s, x) = c(s)x$, for $c : \mathbb{R}_+ \mapsto \mathbb{R}_+$ measurable and deterministic.

We say that a real-valued process $(\zeta_t, t \geq 0)$ has bounded variation, if a.s. for all $T > 0$ the real-valued function $[0, T] \ni t \mapsto \zeta_t$ has bounded variation.

Lemma 2.2. *Let $F : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \mapsto]-1, \infty[$ such that*

- *F is $\mathcal{P} \otimes \mathcal{B}_{\mathbb{R}_+}$ -measurable, where \mathcal{P} denotes the predictable σ -algebra generated by ξ ;*

- there exists a finite constant C_F such that a.s. for almost every $s \geq 0$

$$F(s, 0) = 0, \quad \int \nu(dx) \mathbb{E} \left[\left| F(s, x) \right| \right] \leq C_F < \infty. \quad (2.2)$$

Then

- (1) the process

$$x_t^F := \sum_{s \leq t} F(s, \Delta \xi_s) - \int_0^t ds \int \nu(dx) F(s, x), \quad t \geq 0$$

is a martingale with bounded variation;

- (2) the process

$$\mathcal{E}_t^F := \exp \left(- \int_0^t ds \int \nu(dx) F(s, x) \right) \prod_{s \leq t} \left(1 + F(s, \Delta \xi_s) \right)$$

satisfies

$$\mathcal{E}_t^F = 1 + \int_0^t \mathcal{E}_{s-}^F dx_s^F, \quad t \geq 0. \quad (2.3)$$

Moreover (\mathcal{E}_t^F) is a martingale with bounded variation which satisfies

$$\mathbb{E} \left(\int_0^t |d\mathcal{E}_u^F| \right) \leq 2 C_F t.$$

- (3) for all $t \geq 0$, a.s. $\mathcal{E}_t^F > 0$.

Proof. Notice first that x^F is well defined, since by (2.2)

$$\mathbb{E} \left[\sum_{s \leq t} |F(s, \Delta \xi_s)| \right] = \int_0^t ds \int \nu(dx) \mathbb{E} \left[|F(s, x)| \right] \leq C_F t < \infty.$$

Since $\{(s, \Delta \xi_s), s \geq 0\}$ is a Poisson point process with intensity measure $ds \nu(dx)$, it follows immediately that x^F is a local martingale. Furthermore, a.s. the paths of x^F have bounded variation, since

$$\mathbb{E} \left(\int_0^t |dx_s^F| \right) \leq 2 C_F t, \quad t \geq 0.$$

Therefore, $(x_t^F, t \geq 0)$ is a true martingale; indeed, for any $t > 0$, $\sup_{s \leq t} |x_s^F| \leq \int_0^t |dx_s^F|$, and therefore

$$\mathbb{E} \left(\sup_{s \leq t} |x_s^F| \right) \leq 2 C_F t, \quad t \geq 0;$$

by Proposition IV.1.7 of [7] we obtain the claim.

Since \mathcal{E}^F is the Doléans exponential associated with the martingale x^F , i.e. it satisfies (2.3), it is clear that \mathcal{E}^F is a local martingale (see chapter 5 of [1]). Moreover, since \mathcal{E}^F is non-negative, then it is a super-martingale and in particular $\mathbb{E}(\mathcal{E}_t^F) \leq \mathbb{E}(\mathcal{E}_0^F) = 1$. Furthermore,

$$\mathbb{E} \left(\int_0^t |d\mathcal{E}_u^F| \right) = \mathbb{E} \left(\int_0^t \mathcal{E}_{u-}^F |dx_u^F| \right) \leq \int_0^t \mathbb{E}(\mathcal{E}_u^F) 2 C_F du \leq 2 C_F t.$$

The same argument as for x^F yields:

$$\mathbb{E} \left(\sup_{s \leq t} |\mathcal{E}_s^F| \right) \leq 2 C_F t, \quad t \geq 0,$$

and therefore \mathcal{E}^F is a martingale.

In order to prove that $\mathcal{E}_t^F > 0$ a.s., by (2.2) it is enough to show that

$$\log \prod_{s \leq t} \left(1 + F(s, \Delta \xi_s) \right) = \sum_{s \leq t} \log \left(1 + F(s, \Delta \xi_s) \right) > -\infty.$$

Since $F(s, \Delta \xi_s) = \Delta x_s^F = x_s^F - x_{s-}^F > -1$, and x^F has a.s. bounded variation, then there is a.s. only a finite number of $s \in [0, t]$ such that $\Delta x_s^F < -1/2$ and therefore a.s. $\inf_{s \leq t} \Delta x_s^F =: C_t > -1$. It follows that

$$\sum_{s \leq t} \log \left(1 + \Delta x_s^F \right) \geq -\frac{1}{C_t + 1} \sum_{s \leq t} |\Delta x_s^F| = -\frac{1}{C_t + 1} \int_0^t |dx_u^F| > -\infty, \quad \text{a.s.} \quad \square$$

The main steps in the proofs of Proposition 2.1 and Theorem 1.2 are the estimate (2.5) and the identity (2.6) below, which allow to apply Lemma 2.2 to

$$F(s, 0) := 0, \quad F(s, x) := h(s, x) \cdot \frac{g(H(s, x))}{g(x)} \cdot e^{-\lambda_s H(s, x)} - 1, \quad x > 0. \quad (2.4)$$

Lemma 2.3. *Let $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a C^1 function such that $\phi(0) = 0$,*

$$0 < \kappa^{-1} \leq \phi'(x) \leq \kappa < \infty, \quad |\phi'(x) - \phi'(y)| \leq \kappa |x - y|^\alpha, \quad \forall x, y \in \mathbb{R}_+,$$

where $\kappa > 0$ and $\alpha \in]0, 1[$. We set for all $a \geq 0$

$$F_{a, \phi} : (0, \infty) \mapsto \mathbb{R}, \quad F_{a, \phi} := \phi' \cdot \frac{g(\phi)}{g} \cdot e^{-a\phi} - 1.$$

There exists a finite constant $C = C(\kappa, \alpha, a)$ such that

$$\int_0^\infty |F_{a, \phi}(x)| g(x) dx \leq C, \quad (2.5)$$

and

$$\int_0^\infty F_{a, \phi}(x) g(x) dx = -\Psi(a) - g_0 \log \phi'(0). \quad (2.6)$$

Proof. Notice that $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a diffeomorphism. First we have

$$\begin{aligned} \int_{\kappa^{-1}}^\infty |F_{a, \phi}| g dx &\leq \int_{\kappa^{-1}}^\infty \phi' g(\phi) dx + \int_{\kappa^{-1}}^\infty g dx \\ &= \int_{\phi(\kappa^{-1})}^\infty g(y) dy + \int_{\kappa^{-1}}^\infty g dx \leq 2 \int_{\kappa^{-2}}^\infty g(x) dx < \infty. \end{aligned}$$

Now

$$\begin{aligned} \int_0^{\kappa^{-1}} |F_{a,\phi}| g dx &= \int_0^{\kappa^{-1}} |\phi' g(\phi) e^{-a\phi} - g| dx \\ &\leq \int_0^{\kappa^{-1}} \phi' g(\phi) (1 - e^{-a\phi}) dx + \int_0^{\kappa^{-1}} \phi' \left| g(\phi) - \frac{g_0}{\phi} \right| dx + \int_0^1 g_0 \left| \frac{\phi'(x)}{\phi(x)} - \frac{1}{x} \right| dx \\ &\quad + \int_0^{\kappa^{-1}} \left| \frac{g_0}{x} - g(x) \right| dx + \int_0^{\kappa^{-1}} g(x) (1 - e^{-ax}) dx =: I_0 + I_1 + I_2 + I_3 + I_4. \end{aligned}$$

First we estimate I_2 .

$$\begin{aligned} I_2 &= \int_0^1 g_0 \left| \frac{\phi'(x)}{\phi(x)} - \frac{1}{x} \right| dx = g_0 \int_0^1 \left| \frac{\phi(x) - x\phi'(x)}{x\phi(x)} \right| dx \\ &\leq \int_0^1 \frac{g_0}{\kappa^{-1}x^2} \left| \int_0^x [\phi'(y) - \phi'(x)] dy \right| dx \leq g_0 \kappa^2 \int_0^1 \frac{1}{x^2} \int_0^x y^\alpha dy dx \leq \frac{g_0 \kappa^2}{\alpha(1+\alpha)}. \end{aligned}$$

Recall now that $g(x) = \frac{g_0}{x} + \zeta(x)$ by (H2) above. Then I_3 and I_4 can be estimated by

$$I_3 = \int_0^{\kappa^{-1}} \left| \frac{g_0}{x} - g(x) \right| dx \leq \int_0^1 |\zeta| dx,$$

and

$$I_4 = \int_0^{\kappa^{-1}} g(x) (1 - e^{-ax}) dx \leq \int_0^1 g_0 ax dx + \int_0^1 |\zeta| dx \leq ag_0 + \int_0^1 |\zeta| dx.$$

Then I_0 and I_1 can be estimated similarly by changing variable

$$I_1 = \int_0^{\kappa^{-1}} \phi' \left| g(\phi) - \frac{g_0}{\phi} \right| dx = \int_0^{\phi(\kappa^{-1})} \left| g(x) - \frac{g_0}{x} \right| dx \leq \int_0^1 |\zeta| dx,$$

and

$$I_0 = \int_0^{\kappa^{-1}} \phi' g(\phi) (1 - e^{-a\phi}) dx = \int_0^{\phi(\kappa^{-1})} g(x) (1 - e^{-ax}) dx \leq ag_0 + \int_0^1 |\zeta| dx,$$

since $\phi(\kappa^{-1}) \leq 1$. Therefore, we have obtained

$$\int_0^\infty |F_{a,\phi}| g dx \leq 2ag_0 + \frac{g_0 \kappa^2}{\alpha(1+\alpha)} + 2 \int_{\kappa^{-1}}^\infty g(y) dy + 4 \int_0^1 |\zeta(x)| dx,$$

and (2.5) is proven.

We turn now to the proof of (2.6). By (2.5) and dominated convergence

$$\int_0^\infty F_{a,\phi} g dx = \lim_{\varepsilon \searrow 0} \int_\varepsilon^\infty F_{a,\phi} g dx.$$

For all $\varepsilon > 0$ we have

$$\int_\varepsilon^\infty \phi' g(\phi) e^{-a\phi} dx = \left[y = \phi(x) \right] = \int_{\phi(\varepsilon)}^\infty g(y) e^{-ay} dy.$$

Then we want to compute the limit as $\varepsilon \searrow 0$ of

$$\begin{aligned} \int_{\varepsilon}^{\infty} F_{a,\phi} g \, dx &= \int_{\phi(\varepsilon)}^{\infty} g(x) e^{-ax} \, dx - \int_{\varepsilon}^{\infty} g(x) \, dx \\ &= \int_{\phi(\varepsilon)}^{\infty} g(x) (e^{-ax} - 1) \, dx + \int_{\phi(\varepsilon)}^1 g(x) \, dx - \int_{\varepsilon}^1 g(x) \, dx. \end{aligned}$$

Clearly, by assumptions (H1)-(H2) and by dominated convergence

$$\lim_{\varepsilon \searrow 0} \int_{\phi(\varepsilon)}^{\infty} g(x) (e^{-ax} - 1) \, dx = \int_0^{\infty} g(x) (e^{-ax} - 1) \, dx = -\Psi(a).$$

Now, by assumption (H2)

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \left[\int_{\phi(\varepsilon)}^1 g(x) \, dx - \int_{\varepsilon}^1 g(x) \, dx \right] &= g_0 \lim_{\varepsilon \searrow 0} \left[\int_{\phi(\varepsilon)}^1 \frac{1}{x} \, dx - \int_{\varepsilon}^1 \frac{1}{x} \, dx \right] \\ &= g_0 \lim_{\varepsilon \searrow 0} \log \frac{\varepsilon}{\phi(\varepsilon)} = -g_0 \log \phi'(0). \end{aligned}$$

Then we have obtained (2.6). \square

Proof of Proposition 2.1. It is enough to apply the results of Lemma 2.2 and Lemma 2.3 to $\phi(x) := H(s, x)$, $a = \lambda_s$ and F defined in (2.4). Positivity of $M_t^{H,\lambda}$ follows from point (3) of Lemma 2.2. \square

Proof of Theorem 1.2. Notice that $M^H = M^{H,\lambda}$ for $\lambda \equiv 0$. By Proposition 2.1, M^H is a martingale with expectation 1. Then, for any bounded measurable $\lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$, by Proposition 2.1 we obtain

$$\mathbb{E} \left(\exp \left(- \int_0^t \lambda_s \, d\xi_s^H \right) M_t^H \right) = \exp \left(- \int_0^t \Psi(\lambda_s) \, ds \right), \quad t \geq 0.$$

The desired result now follows by uniqueness of the Laplace transform. \square

3. QUASI-INVARIANCE PROPERTIES OF (\mathcal{L}) -SUBORDINATORS

In this section we point out two special cases of Theorem 1.2. We consider a measurable function $k : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ which satisfies, for some finite constants $\kappa \geq 1$ and $\alpha \in]0, 1[$

$$|k(s, x) - k(s, y)| \leq \kappa |x - y|^\alpha, \quad \forall s, x, y \in \mathbb{R}_+,$$

$$0 < \kappa^{-1} \leq k(s, x) \leq \kappa < \infty, \quad \forall s, x \in \mathbb{R}_+,$$

and we set

$$K(s, x) := \int_0^x k(s, y) \, dy, \quad \forall x, s \geq 0,$$

Notice that $K(s, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is necessarily bijective for all $s \geq 0$.

3.1. Quasi-invariance of ξ under composition with a diffeomorphism.

$$H(s, x) := K(s, \xi_{s-} + x) - K(s, \xi_{s-}), \quad h(s, x) := k(s, \xi_{s-} + x), \quad s, x \geq 0,$$

we find that

$$\xi_t^H = \sum_{s \leq t} H(s, \Delta \xi_s) = K(t, \xi_t), \quad t \geq 0.$$

Moreover (1.1) and (1.2) are satisfied and Theorem 1.2 becomes

Corollary 3.1. *The process*

$$G_t^K := \exp \left(g_0 \int_0^t \log k(s, \xi_s) ds \right) \prod_{s \leq t} \left[k(s, \xi_s) \cdot \frac{g(K(s, \xi_s) - K(s, \xi_{s-}))}{g(\Delta \xi_s)} \right], \quad t \geq 0,$$

is a non-negative (\mathcal{F}_t) -martingale with $\mathbb{E}(G_t^K) = 1$ and a.s. $G_t^K > 0$. Then we can define a probability measure \mathbb{P}^K such that $\mathbb{P}_{|\mathcal{F}_t}^K = G_t^K \cdot \mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$. Under \mathbb{P}^K , $(K(t, \xi_t), t \geq 0)$ is distributed as $(\xi_t, t \geq 0)$ under \mathbb{P} .

This result can be interpreted by saying that the law of (ξ_t) is quasi-invariant under (deterministic) non-linear transformations $(\xi_t, t \geq 0) \mapsto (K(t, \xi_t), t \geq 0)$.

3.2. Quasi-invariance of ξ under transformations of jumps.

$$H(s, x) := K(s, x), \quad h(s, x) := k(s, x), \quad s, x \geq 0,$$

we find that (1.1) and (1.2) are satisfied and Theorem 1.2 becomes

Corollary 3.2. *The process*

$$N_t^K := \exp \left(g_0 \int_0^t \log k(s, 0) ds \right) \prod_{s \leq t} \left[k(s, \Delta \xi_s) \cdot \frac{g(K(s, \Delta \xi_s))}{g(\Delta \xi_s)} \right], \quad t \geq 0,$$

is a non-negative (\mathcal{F}_t) -martingale with $\mathbb{E}(N_t^K) = 1$ and a.s. $N_t^K > 0$. Then we can define a probability measure \mathbb{P}^K such that $\mathbb{P}_{|\mathcal{F}_t}^K = N_t^K \cdot \mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$. Under \mathbb{P}^K , the process

$$\xi_t^K = \sum_{s \leq t} K(s, \Delta \xi_s), \quad t \geq 0,$$

is distributed as ξ under \mathbb{P} .

This result can be interpreted by saying that the law of (ξ_t) is quasi-invariant under (deterministic) non-linear transformation of the jumps of ξ : $(\Delta \xi_t, t \geq 0) \mapsto (K(t, \Delta \xi_t), t \geq 0)$.

3.3. Quasi-invariance properties of the gamma process. We now write the results of Corollaries 3.1 and 3.2 in the special case of the gamma process (γ_t) . Here

$$g(x) = \frac{e^{-x}}{x}, \quad x > 0, \quad g_0 = 1, \quad \Psi(\lambda) = \log(1 + \lambda).$$

Corollary 3.3. *We set for all $t \geq 0$*

$$Y_t^K := \exp \left(\gamma_t - K(t, \gamma_t) + \int_0^t \log k(s, \gamma_s) ds \right) \prod_{s \leq t} \left[\frac{k(s, \gamma_s) \cdot \Delta \gamma_s}{K(s, \gamma_s) - K(s, \gamma_{s-})} \right]. \quad (3.1)$$

Then (Y_t^K) is a martingale with $\mathbb{E}(Y_t^K) = 1$ and a.s. $Y_t^K > 0$. Hence, we can define a probability measure \mathbb{P}^K such that $\mathbb{P}_{|\mathcal{F}_t}^K = Y_t^K \cdot \mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$. Under \mathbb{P}^K , $(K(t, \gamma_t), t \geq 0)$ is distributed as $(\gamma_t, t \geq 0)$ under \mathbb{P} .

Corollary 3.4. *The process*

$$Z_t^K := \exp \left(\gamma_t - \sum_{s \leq t} K(s, \Delta \gamma_s) + \int_0^t \log k(s, 0) ds \right) \prod_{s \leq t} \left[k(s, \Delta \gamma_s) \cdot \frac{\Delta \gamma_s}{K(s, \Delta \gamma_s)} \right],$$

$t \geq 0$, is a non-negative (\mathcal{F}_t) -martingale with $\mathbb{E}(Z_t^K) = 1$ and a.s. $Z_t^K > 0$. Then we can define a probability measure \mathbb{P}^K such that $\mathbb{P}_{|\mathcal{F}_t}^K = Z_t^K \cdot \mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$. Under \mathbb{P}^K , the process

$$\gamma_t^K = \sum_{s \leq t} K(s, \Delta \gamma_s), \quad t \geq 0,$$

is distributed as $(\gamma_t, t \geq 0)$ under \mathbb{P} .

4. QUASI-INVARIANCE PROPERTIES OF THE DIRICHLET PROCESS

We fix $T > 0$ and we denote by $(D_t^{(T)} : t \in [0, T])$ the Dirichlet process over the time interval $[0, T]$, i.e. $D_t^{(T)} := \gamma_t / \gamma_T$ where (γ_t) is a gamma process. Since T is fixed we omit the superscript (T) .

We consider a measurable function $k : [0, T] \times [0, 1] \mapsto [0, 1]$ which satisfies, for some finite constants $\kappa \geq 1$ and $\alpha \in]0, 1[$

$$\begin{aligned} |k(s, x) - k(s, y)| &\leq \kappa |x - y|^\alpha, & \forall x, y \in [0, 1], s \in [0, T], \\ 0 < \kappa^{-1} &\leq k(s, x) \leq \kappa < \infty, & \forall x \in [0, 1], s \in [0, T], \end{aligned}$$

and we set

$$K(s, x) := \int_0^x k(s, y) dy, \quad \forall x \in [0, 1], s \in [0, 1].$$

4.1. Quasi-invariance of D under composition with a diffeomorphism. We want to give a martingale proof of a relation originally obtained by von Renesse-Sturm in [6]. In this subsection we suppose that k also satisfies

$$\int_0^1 k(s, y) dy = 1, \quad \forall s \in [0, T],$$

so that

$$K(s, 0) = 0, \quad K(s, 1) = 1, \quad \forall s \in [0, 1].$$

Notice that $K(s, \cdot) : [0, 1] \mapsto [0, 1]$ is necessarily bijective for all $s \in [0, T]$. We set for $t < T$

$$L_t^{K,T} := \left(\frac{1 - K(t, D_t)}{1 - D_t} \right)^{T-t-1} \exp \left(\int_0^t \log k(s, D_s) ds \right) \prod_{s \leq t} \left[\frac{k(s, D_s) \cdot \Delta D_s}{K(s, D_s) - K(s, D_{s-})} \right],$$

$$L_T^{K,T} := \frac{1}{k(T,1)} \exp \left(\int_0^T \log k(s, D_s) ds \right) \prod_{s \leq T} \left[\frac{k(s, D_s) \cdot \Delta D_s}{K(s, D_s) - K(s, D_{s-})} \right].$$

Theorem 4.1.

- (1) $(L_t^{K,T}, t \in [0, T])$ is a martingale with respect to the natural filtration of D , such that $\mathbb{E}(L_t^{K,T}) = 1$ and a.s. $L_t^{K,T} > 0$, for all $t \in [0, T]$.
- (2) Under $\mathbb{P}^{K,T} := L_T^{K,T} \cdot \mathbb{P}$ the process $(K(s, D_s), s \in [0, T])$ has the same law as $(D_s, s \in [0, T])$ under \mathbb{P} .

This theorem gives quasi-invariance of the law of D under non-linear transformations $(D_s, s \in [0, T]) \mapsto (K(s, D_s), s \in [0, T])$.

Proof of Theorem 4.1. Let first $t < T$. By the Markov property, for all bounded Borel $\Phi : \mathcal{D}([0, t]) \mapsto \mathbb{R}_+$

$$\begin{aligned} \mathbb{E}(\Phi(D_s, s \leq t)) &= \mathbb{E} \left(\Phi(\gamma_s, s \leq t) 1_{(\gamma_t < 1)} \frac{p_{T-t}(1 - \gamma_t)}{p_T(1)} \right) \\ &= \mathbb{E} \left(\Phi(\gamma_s, s \leq t) 1_{(\gamma_t < 1)} (1 - \gamma_t)^{T-t-1} e^{\gamma_t} \right) \frac{\Gamma(T)}{\Gamma(T-t)}. \end{aligned}$$

Consider the following extension of K to $[0, T] \times \mathbb{R}_+$, that we still call K

$$K(s, x) := K(s, x) 1_{(x \leq 1)} + k(s, 1)(x - 1) 1_{(x > 1)}, \quad x \geq 0, \quad s \in [0, T].$$

Let us consider the process (Y_t^K) as defined in (3.1). Notice that $K(t, \cdot)$ is strictly increasing, so that $K(t, \gamma_t) < 1$ iff $\gamma_t < 1$. Then, for all bounded Borel $\Phi : \mathcal{D}([0, t]) \mapsto \mathbb{R}_+$, $t < T$, by Corollary 3.3

$$\begin{aligned} \mathbb{E} \left(\Phi(K(\cdot, D_s)) L_t^{K,T} \right) &= \frac{\Gamma(T)}{\Gamma(T-t)} \mathbb{E} \left(1_{(\gamma_t < 1)} (1 - K(t, \gamma_t))^{T-t-1} \Phi(K(\cdot, \gamma_s)) e^{K(t, \gamma_t)} Y_t^K \right) \\ &= \frac{\Gamma(T)}{\Gamma(T-t)} \mathbb{E} \left(1_{(\gamma_t < 1)} (1 - \gamma_t)^{T-t-1} \Phi(\gamma_s) e^{\gamma_t} \right) = \mathbb{E}(\Phi(D_s)), \end{aligned}$$

and this concludes the proof for $t < T$.

We consider now the case $t = T$. For all bounded Borel $\Phi : \mathcal{D}([0, T]) \mapsto \mathbb{R}_+$ and $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, by Corollary 3.3

$$\mathbb{E}(\Phi(K(s, \gamma_s), s \in [0, T]) \varphi(K(T, \gamma_T)) Y_T^K) = \mathbb{E}(\Phi(\gamma_s, s \in [0, T]) \varphi(\gamma_T)). \quad (4.1)$$

We set for all $x > 0$

$$Y_T^{K,x} := \exp \left(x - K(T, x) + \int_0^T \log k(s, x D_s) ds \right) \prod_{s \leq T} \left[\frac{k(s, x D_s) \cdot x \Delta D_s}{K(s, x D_s) - K(s, x D_{s-})} \right].$$

In the right hand side of (4.1) we condition on the value of γ_T , obtaining

$$\mathbb{E}(\Phi(\gamma_s) \varphi(\gamma_T)) = \int_0^\infty p_T(y) \mathbb{E}(\Phi(y D_s)) \varphi(y) dy.$$

In the left hand side of (4.1), conditioning on the value of γ_T , we obtain

$$\mathbb{E}(\Phi(K(\cdot, \gamma_s)) \varphi(K(T, \gamma_T)) Y_T^K) = \int_0^\infty p_T(x) \mathbb{E}(\Phi(K(\cdot, x D_s)) Y_T^{K,x}) \varphi(K(T, x)) dx.$$

In order to compare this expression with the one above for the right hand side, we use the change of variable $x = K(T, y)$. To this aim, we denote by $C : \mathbb{R}_+ \mapsto \mathbb{R}_+$ the inverse of $K(T, \cdot)$, i.e. we suppose that $K(T, C(x)) = x$ for all $x \geq 0$. Then we have

$$\begin{aligned} \mathbb{E} \left(\Phi(K(\cdot, \gamma_T)) \varphi(K(T, \gamma_T)) Y_T^K \right) &= \int_0^\infty p_T(x) \mathbb{E} \left(\Phi(K(\cdot, xD)) Y_T^{K,x} \right) \varphi(K(T, x)) dx \\ &= \int_0^\infty p_T(C(y)) \mathbb{E} \left(\Phi(K(\cdot, C(y)D)) Y_T^{K,C(y)} \right) \varphi(y) C'(y) dy. \end{aligned}$$

Since this is true for any bounded measurable $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, we obtain for all $y > 0$

$$\frac{p_T(C(y)) C'(y)}{p_T(y)} \mathbb{E} \left(\Phi(K(\cdot, C(y)D)) Y_T^{K,C(y)} \right) = \mathbb{E} (\Phi(yD)).$$

For $y = 1$, since $K(T, 1) = 1 = C(1)$, we obtain the desired result

$$\mathbb{E} \left(\Phi(K(\cdot, D)) L_T^{K,T} \right) = \mathbb{E} (\Phi(D)). \quad \square$$

Remark 4.2. Von Renesse-Sturm prove the second result of Theorem 4.1 in [6]. The proof there hinges on explicit computations related to the finite-dimensional distributions of D .

4.2. Quasi-invariance of D under transformation of the jumps. Again, we consider the Dirichlet process $(D_t^{(T)}, t \in [0, T])$, and we drop the superscript (T) , since T is fixed. We set

$$\Delta D_s := D_s - D_{s-}, \quad D_t^K := \frac{\sum_{s \leq t} K(s, \Delta D_s)}{\sum_{s \leq T} K(s, \Delta D_s)}, \quad t \in [0, T].$$

Theorem 4.3. *The laws of $(D_t^K, t \in [0, T])$ and $(D_t, t \in [0, T])$ are equivalent.*

In the proof of Theorem 4.3 we also compute explicitly the Radon-Nikodym density. Handa [5] considers the particular case $K(s, x) = c(s)x$, where $c : [0, T] \mapsto \mathbb{R}_+$ is measurable.

Proof. We set

$$\gamma_t^K := \sum_{s \leq t} K(s, \Delta \gamma_s), \quad t \geq 0.$$

Since $(D_t, t \in [0, T])$ is a gamma bridge, then the law of $(D_t^K, t \in [0, T])$ coincides with the law of $(\gamma_t^K / \gamma_T^K, t \in [0, T])$ under the conditioning $\{\gamma_T = 1\}$.

We define $J : [0, T] \times \mathbb{R}_+ \mapsto \mathbb{R}$, such that, for all $s \in [0, T]$, $J(s, K(s, x)) = x$ for all $x \geq 0$. In other words, $J(s, \cdot)$ is the inverse of $K(s, \cdot)$. In particular, notice that

$$(\gamma^K)^J = \gamma.$$

By Corollary 3.4, for all $\Phi : \mathcal{D}([0, T]) \mapsto \mathbb{R}$ bounded and Borel

$$\mathbb{E} \left(\Phi \left(\frac{\gamma_s^K}{\gamma_T^K}, s \leq T \right) \varphi(\gamma_T) Z_T^K \right) = \mathbb{E} \left(\Phi \left(\frac{\gamma_s}{\gamma_T}, s \leq T \right) \varphi(\gamma_T^J) \right). \quad (4.2)$$

Notice that

$$\gamma_T^J = \sum_{s \leq T} J(s, \Delta \gamma_s) = \sum_{s \leq T} J(s, \gamma_T \cdot \Delta D_s) = \psi_D(\gamma_T),$$

where $D_t := \gamma_t/\gamma_T$, $t \in [0, T]$, is independent of γ_T and

$$\psi_D(x) := \sum_{s \leq T} J(s, x \cdot \Delta D_s), \quad x \geq 0.$$

Notice that $\psi_D : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is C^1 and by dominated convergence

$$\psi'_D(x) = \sum_{s \leq T} \Delta D_s \cdot J(s, x \cdot \Delta D_s) \geq \kappa^{-1} > 0, \quad \forall x \geq 0,$$

since $\Delta D_s \geq 0$ and $\sum_{s \leq T} \Delta D_s = 1$. Also by dominated convergence, ψ'_D is continuous. Therefore $\psi_D : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is invertible, with C^1 inverse $\zeta_D := \psi_D^{-1}$. In the sequel, We may write ζ_D^K for ζ_D , in order to stress that it also depends on K . Then, for all $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}$ bounded and Borel, we obtain by (4.2)

$$\begin{aligned} \mathbb{E} \left(\Phi \left(\frac{\gamma_s^K}{\gamma_T^K}, s \leq T \right) \varphi(\gamma_T) Z_T^K \right) &= \mathbb{E} (\Phi(D_s, s \leq T) \varphi(\psi_D(\gamma_T))) \\ &= \mathbb{E} \left(\Phi(D_s, s \leq T) \int_0^\infty p_T(y) \varphi(\psi_D(y)) dy \right) = [x = \psi_D(y)] \\ &= \int_0^\infty \varphi(x) \mathbb{E} (\Phi(D_s, s \leq T) p_T(\zeta_D(x)) \zeta'_D(x)) dx. \end{aligned} \quad (4.3)$$

Now, setting for all $t \in [0, T]$

$$D_t^{K,x} := \frac{\sum_{s \leq t} K(s, x \cdot \Delta D_s)}{\sum_{s \leq T} K(s, x \cdot \Delta D_s)},$$

$$U_T^{K,x} := \exp \left(x - \sum_{s \leq T} K(s, x \Delta D_s) + \int_0^T \log k(s, 0) ds \right) \prod_{s \leq T} \left[\frac{k(s, x \Delta D_s) \cdot x \Delta D_s}{K(s, x \Delta D_s)} \right],$$

then we have

$$\mathbb{E} \left(\Phi \left(\frac{\gamma_s^K}{\gamma_T^K}, s \leq T \right) \varphi(\gamma_T) Z_T^K \right) = \int_0^\infty \varphi(x) \mathbb{E} \left(\Phi(D_s^{K,x}, s \leq T) \cdot U_T^{K,x} \right) p_T(x) dx. \quad (4.4)$$

Since $D^{K,1} = D^K$, setting

$$U_T^K := U_T^{K,1} = \exp \left(1 - \sum_{s \leq T} K(s, \Delta D_s) + \int_0^T \log k(s, 0) ds \right) \prod_{s \leq T} \left[\frac{k(s, \Delta D_s) \cdot \Delta D_s}{K(s, \Delta D_s)} \right],$$

we obtain by (4.3) and (4.4) for $x = 1$

$$\mathbb{E} (\Phi(D_s^K, s \leq T) \cdot U_T^K) = \mathbb{E} \left(\Phi(D_s, s \leq T) \frac{p_T(\zeta_D^K(1))}{p_T(1)} (\zeta_D^K)'(1) \right). \quad \square$$

5. STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY (\mathcal{L}) -SUBORDINATORS

In this section we give an application to stochastic differential equations driven by a (\mathcal{L}) -subordinator ξ . See [2] for a survey of SDEs driven by Lévy processes.

We consider the SDE

$$dX_t = m(t, X_{t-}) d\xi_t, \quad X_0 = 0, \quad (5.1)$$

where

- (1) $m : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto (0, +\infty)$ is measurable;
- (2) m and $1/m$ are bounded
- (3) $\mathbb{R}_+ \ni a \mapsto m(s, a)$ is Lipschitz, uniformly in $s \geq 0$.

Then we have

Theorem 5.1. *There exists a pathwise-unique solution of (5.1) and the law of (X, ξ) under \mathbb{P} coincides with the law of (ξ, ξ^H) under \mathbb{P}^H , where*

$$H(s, x) := \frac{x}{m(s, \xi_{s-})}, \quad s \geq 0, \ x \geq 0. \quad (5.2)$$

Proof. Let $T > 0$ and denote by $\mathcal{I}([0, T])$ the set of all bounded increasing functions $\omega : [0, T] \mapsto \mathbb{R}_+$. We define the map $\Lambda_T : \mathcal{I}([0, T]) \mapsto \mathcal{I}([0, T])$

$$\Lambda_T(\omega)(t) := \int_0^t m(s, \omega_{s-}) d\xi_s, \quad t \in [0, T].$$

For L large enough, Λ_T is a contraction in $\mathcal{I}([0, T])$ with respect to the metric

$$d_L(\omega, \omega') := \sup_{t \in [0, T]} e^{-Lt} |\omega_t - \omega'_t|,$$

and the solution of (5.1) on the time interval $[0, T]$ is the unique fixed point X of Λ_T . Moreover, there exists a measurable map $W_T : \mathcal{I}([0, T]) \mapsto \mathcal{I}([0, T])$, such that $X = W_T(\xi_{|[0, T]})$.

Let us define H as in (5.2), and set ξ^H as in (1.4)

$$\xi_t^H := \sum_{s \leq t} H(s, \Delta \xi_s) = \int_0^t \frac{1}{m(s, \xi_{s-})} d\xi_s$$

Note that

$$d\xi_t^H = \frac{1}{m(t, \xi_{t-})} d\xi_t \implies d\xi_t = m(t, \xi_{t-}) d\xi_t^H.$$

Then, $\xi_{|[0, T]} = W_T(\xi_{|[0, T]}^H)$ for any $T > 0$. On the other hand, by Theorem 1.2, ξ^H under \mathbb{P}^H has the same law as ξ under \mathbb{P} , and this concludes the proof. \square

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